# GUIDELINES AND PRACTICE PROBLEMS FOR EXAM 1

Exam 1 will cover all material from the first day of class up to and including whatever we cover on Thursday February 12. Questions on the exam will be of the following types: Stating definitions, propositions or theorems; short answer; true-false; and presentation of a proof of a theorem. I will try to keep time-consuming calculations to a minimum. For example, I may give you vectors $v_1, v_2, v_3$ in $\mathbb{R}^4$ and ask how you would determine if a vector $u$ was in the span of those vectors. Rather than working out the details, you would describe how to set up a system of linear equations that would determine if $u$ is in $\text{Span}\{v_1, v_2, v_3\}$. Or similar questions relating to linear independence, bases, linear transformations and matrix calculations.

Any definitions, propositions theorems, corollaries that you need to know how to state appear in the Daily Update. You will need to be able to answer brief questions about these results as well as true-false statements about these results. Most of the definitions you need to know are also in the Daily Update, but it is best to check your notes for all definitions we have given by February 20.

You will also be responsible for working ay any type of problem that was previously assigned as homework.

On the Exam you will be required to state and provide a proof of one of the following Theorems.

(i) The Exchange Theorem. You must state the theorem in its full generality and provide a proof that given a spanning set with four vectors, spanning the subspace $W$, and any independent set of two vectors in $W$, two of the vectors in the spanning set can be replaced by the given independent vectors and the new set still spans $W$.

(ii) If $V$ is a finite dimensional vector space, then any linearly independent set in $V$can be extended to a basis for $V$.

(iii) Define elementary $2 \times 2$ matrices and use elementary matrices to proof that $|AB| = |A| \cdot |B|$ for $2 \times 2$ matrices $A$ and $B$ such that $B$ is invertible.

## Practice Problems

The following practice problems are from our textbook.

Section 1.3: 9b, c, d. Section 1.5: Find a basis for the solutions space of the systems of equations in 1a,c,f; 3c, 4b, 5a

Section 1.6: 2d, 4, 7c,

Section 3.2: 1b, 2b, 4b

Section 3.3: 1a (Use the adjoint to find the inverse), 2b, 7b

Section 4.1: 3d, e (find eigenvalues, and bases for the corresponding eigenvectors for the given matrices. Ignore the linear mappings).

# Selected Solutions to Exam 1 Practice Problems

## Section 1.3

**#9b** If $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W_1 + W_2$ if we can write $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = w_1 + w_2$ with $w_1 \in W_1$ and $w_2 \in W_2$. I.e.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}}_{\text{in } W_1} + \underbrace{\left\{ \beta \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}}_{\text{in } W_2}$$

i.e. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ for some $\alpha, \beta, \gamma \in \mathbb{R}$

Gauss Elim.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 1 & 0 & 1 & y \\ 1 & 0 & 0 & z \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & z \\ 1 & 0 & 1 & y \\ 1 & 1 & 1 & x \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & z \\ 0 & 0 & 1 & y-z \\ 0 & 1 & 1 & x \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & z \\ 0 & 0 & 1 & y-z \\ 0 & 1 & 0 & x-y+z \end{array}\right) \Rightarrow \alpha = z, \ \beta = x-y+z, \ \gamma = y-z$$

Since this system has a unique sol$^{\underline{n}}$ $\Rightarrow$ system has a unique sol$^{\underline{n}}$ when $x=y=z=0$ i.e. $w_1 + w_2 = \vec{0}$ $\Rightarrow w_1 = \vec{0} = w_2 \Rightarrow \mathbb{R}^3 = W_1 \oplus W_2$ (See part (f))

**#9c** Note $F(x) = f(x) + f(-x)$ is even and $f(x) - f(-x) = G(x)$ is odd e.g. $F(-x) = f(-x) + f(--x) = f(x) + f(x) = F(x)$.

Thus $F(R) = W_1 + W_2$, since $f(x) = \frac{1}{2}F(x) + \frac{1}{2}G(x)$

Suppose $g(x) \in W_1 \cap W_2 \Rightarrow g(x)$ is even and odd $\Rightarrow g(x) = g(-x)$ and $-g(x) = g(-x)$

$\therefore\ g(x) = -g(x) \Rightarrow g(x) = 0 \Rightarrow W_1 \cap W_2 = \vec{0}$

$\Rightarrow F(\mathbb{R}) = W_1 \oplus W_2$

9d) Take $v \in V$. If $v = w_1 + w_2$ and $v = w_1' + w_2'$

$\Rightarrow \underbrace{w_1 - w_1'}_{\text{in } W_1} = \underbrace{w_2' - w_2}_{\text{in } W_2} \Rightarrow$ in $W_1 \cap W_2 = \vec{0}$

$\Rightarrow w_1 - w_1' = \vec{0} \Rightarrow w_1 = w_1'$

$w_2 - w_2' = \vec{0} \Rightarrow w_2 = w_2'$

Section 1.5: 1 (a) back of book

1 c) $\begin{pmatrix} -1/3 \\ 1/9 \\ 1 \end{pmatrix}$

1 f) Must take homogeneous system to get a subspace.

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 0 \end{array}\right)$$

$$\rightarrow \cancel{\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)} \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 1/6 & 1/2 & 0 \end{array}\right) \Leftrightarrow \begin{array}{l} x_1 = 1/3\, x_4 \\ x_3 = -1/6\, x_4 - 1/2\, x_5 \end{array}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} x_4 \\ x_2 \\ -\frac{1}{6} x_4 - \frac{1}{2} x_5 \\ x_4 \\ x_5 \end{pmatrix}$$

$$= x_4 \begin{pmatrix} 1/3 \\ 0 \\ -1/6 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ -1/2 \\ 0 \\ 1 \end{pmatrix}$$

Basis

Section 1.6 : 2d $\quad p(x) = ax^3 + bx^2 + cx + d$ = Typical element in $P_3(\mathbb{R})$. Suppose $p(2) = 0 = p(-1)$

$$\Rightarrow \begin{array}{l} 8a + 4b + 2c + d = 0 \\ -a + b - c + d = 0 \end{array} \Rightarrow \text{Solve}$$

$$\left(\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 0 \\ 8 & 4 & 2 & 1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 0 & 12 & -6 & 9 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & \frac{3}{4} & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1/2 & -1/4 & 0 \\ 0 & 1 & -1/2 & 3/4 & 0 \end{array}\right)$$

$$a = -\tfrac{1}{2}c + \tfrac{1}{4}d$$

$$b = \tfrac{1}{2}c - 3/4\, d$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}c + \frac{1}{4}d \\ \frac{1}{2}c - \frac{3}{4}d \\ c \\ d \end{pmatrix}$$

$$= c \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} \frac{1}{4} \\ -\frac{3}{4} \\ 0 \\ 1 \end{pmatrix}$$

basis

$$\left.\begin{aligned} \therefore\ p_1(x) &= -\tfrac{1}{2}x^3 + \tfrac{1}{2}x^2 + 1\cdot x + 0 \\ p_2(x) &= \tfrac{1}{4}x^3 - \tfrac{3}{4}x^2 + 0\cdot x + 1 \end{aligned}\right\} \text{basis}$$

4. $\dim W_1 = \dim W_2 = 2.$

Label the given vectors $w_1, w_1'$; $w_2, w_2'$.
Then $W_1 + W_2 =$ space spanned by $w_1, w_1', w_2, w_2'$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 1 \\ -4 & -2 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -6 & -3 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & 6 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 10 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 7/10 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \text{basis for } W_1 + W_2$$

$\Rightarrow \dim W_1 + W_2 = 3.$

Since $\dim W_1 = 2 = \dim W_2 \Rightarrow \dim W_1 \cap W_2 = 0$ or $1$.

Suppose $aw_1 + bw_1' = cw_2 + dw_2'$ is in $W_1 \cap W_2$

$\Rightarrow aw_1 + bw_1' + cw_2 + dw_2' = \vec{0}$

This $4 \times 4$ system has a non trivial sol$^{\underline{n}}$ since coeff matrix has rank 3

$\therefore W_1 \cap W_2 \neq \vec{0} \Rightarrow \dim W_1 \cap W_2 = 1$

$\therefore 3 = 2 + 2 - 1$

$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

Section 3.2

2b) $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 4 \\ 3 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & -4 & 1 \end{pmatrix}$

$\rightarrow \underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 17 \end{pmatrix}}_{A'} \Rightarrow \det A' = 17 = \det A$ since these EROS do not change det

4b $\det\begin{pmatrix} -1-a & 2 \\ 3 & -a \end{pmatrix} = a^2 + a - 6$

is not zero over $\mathbb{R}$ unless $a = \frac{-1 \pm \sqrt{1 - 4(-6)}}{2}$

$= \frac{-1 \pm 5}{2} = -3$ or $2$

Section 3.3 1(b) $\begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} = -2$

Cofactor matrix $\begin{pmatrix} 0 & -1 \\ -2 & 3 \end{pmatrix}$

Adjoint $= \begin{pmatrix} 0 & -2 \\ -1 & 3 \end{pmatrix}$

Inverse $= -\frac{1}{2} \begin{pmatrix} 0 & -2 \\ -1 & 3 \end{pmatrix}$

7(b): If $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\Rightarrow x = \frac{\begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ 1 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -3 \\ 0 & 1 & 4 \end{vmatrix}} \quad y = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \\ 0 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -3 \\ 0 & 1 & 4 \end{vmatrix}} \quad z = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -3 \\ 0 & 1 & 4 \end{vmatrix}}$$

Section 4.1 3(d) $A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$

$$P_A(x) = \begin{vmatrix} x-1 & 1 & -3 \\ 0 & x-1 & 0 \\ 0 & -4 & x-1 \end{vmatrix} = (x-1)\begin{vmatrix} x-1 & 0 \\ -4 & x-1 \end{vmatrix} = (x-1)^3$$

$\Rightarrow \lambda = 1$ is the only eigenvalue

$E_1$ = nullspace of $\begin{pmatrix} 0 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}$

$\xrightarrow{\text{EROs}} \begin{pmatrix} 0 & 1 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Basis for null space $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ = basis for $E_1$

of $\begin{pmatrix} 0 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}$